4.NF Using Benchmarks to Compare Fractions

Task

Melissa gives her classmates the following explanation for why $\frac{1}{5} < \frac{2}{7}$:

I can compare both $\frac{1}{5}$ and $\frac{2}{7}$ to $\frac{1}{4}$.

Since $\frac{1}{5}$ and $\frac{1}{4}$ are unit fractions and fifths are smaller than fourths, I know that $\frac{1}{5} < \frac{1}{4}$.

I also know that $\frac{1}{4}$ is the same as $\frac{2}{8}$, so $\frac{2}{7}$ is bigger than $\frac{1}{4}$.

Therefore $\frac{1}{5} < \frac{2}{7}$.

a. Explain each step in Melissa's reasoning. Is she correct?

b. Use Melissa's strategy to compare $\frac{29}{60}$ and $\frac{45}{88}$, this time comparing both fractions with $\frac{1}{2}$.

c. Use Melissa's strategy to compare $\frac{8}{25}$ and $\frac{19}{45}$. Explain which fraction you chose for comparison and why.

IM Commentary

This task is intended primarily for instruction. The goal is to provide examples for comparing two fractions, $\frac{1}{5}$ and $\frac{2}{7}$ in this case, by finding a benchmark fraction which lies in between the two. In Melissa's example, she chooses $\frac{1}{4}$ as being larger than $\frac{1}{5}$ and smaller than $\frac{2}{7}$.

This is an important method for comparing fractions and one which requires a strong number sense and ability to make mental calculations. It is, however, a difficult ability to assess because the method is only appropriate when there is a clear benchmark fraction to be used. In part (c) of the problem, for example, students may see the denominator of 25 and think that $\frac{1}{5}$ or $\frac{2}{5}$ would be potential fractions to use for comparison. In this case, it turns out that $\frac{2}{5}$ is an excellent choice which works well. However, if the numbers were different (for example $\frac{8}{25}$ and $\frac{14}{39}$) then there may be no fifths between them and students might spend a lot of time spinning their wheels trying to make $\frac{1}{5}$ or $\frac{2}{5}$ work. In addition to $\frac{2}{5}$, suggested by the denominator 25, both fractions are less than $\frac{1}{2}$, so identifying $\frac{1}{3}$ as a possibility for comparison might also come from the students and could be suggested if they struggle.

The Standards for Mathematical Practice focus on the nature of the learning experiences by attending to the thinking processes and habits of mind that students need to develop in order to attain a deep and flexible understanding of mathematics. Certain tasks lend themselves to the demonstration of specific practices by students. The practices that are observable during exploration of a task depend on how instruction unfolds in the classroom. While it is possible that tasks may be connected to several practices, only one practice connection will be discussed in depth. Possible secondary practice connections may be discussed but not in the same degree of detail.

This particular task is linked very intentionally to Mathematical Practice Standard 3, critique the reasoning of others. Students are asked to explain and critique the reasoning of their classmate, Melissa. This type of task provides students with an opportunity to distinguish a reasonable explanation from that which is flawed. If there is a flaw in the argument they can further explain why it is flawed. Learning how to argue whether a claim is true or false concisely and precisely becomes a routine part of a student's mathematical work. This task is further extended by directing students to

explore Melissa's strategy with 2 additional examples. Their exploration may spark a conversation about when this strategy is most effective and what other strategies may be more effective and why. (MP.5)

Solution

a. Melissa's reasoning is correct. For the first step $\frac{1}{5}$ represents one of five equal pieces that make up a whole. $\frac{1}{4}$ represents one of four equal pieces making up the *same* whole. Since there are fewer of the equal pieces of size $\frac{1}{4}$ making up the same whole, $\frac{1}{5} < \frac{1}{4}$.

Next, Melissa argues that $\frac{1}{4} < \frac{2}{7}$. To compare these two fractions, she first changes the denominator of $\frac{1}{4}$ from 4 to 8. To write $\frac{1}{4}$ as a fraction with 8 in the denominator means that the denominator is multiplied by 2. Multiplying the numerator by 2 also gives

$$\frac{1}{4} = \frac{2 \times 1}{2 \times 4} = \frac{2}{8}.$$

Now $\frac{2}{8} < \frac{2}{7}$ because $\frac{2}{8}$ represents two of eight equal pieces which make up a whole while $\frac{2}{7}$ represents two of seven equal pieces that make up the *same* whole. Since there are fewer of the equal pieces of size $\frac{1}{7}$ making up the same whole, $\frac{2}{8} < \frac{2}{7}$.

Combining the work from the first two paragraphs gives

$$\frac{1}{5} < \frac{1}{4} < \frac{2}{7}$$

and so $\frac{1}{5} < \frac{2}{7}$. Melissa's reasoning is involved but correct.

b. Using Melissa's strategy, the goal is to compare $\frac{29}{60}$ to $\frac{1}{2}$ and then to compare $\frac{45}{88}$ to $\frac{1}{2}$. For $\frac{29}{60}$ and $\frac{1}{2}$ we can compare these fractions by finding a common denominator. Since 2 is a factor of 60 we can use 60 as a common denominator. To write $\frac{1}{2}$ with a denominator of 60 we need to multiply the denominator (and numerator) by 30:

$$\frac{1}{2} = \frac{30 \times 1}{30 \times 2} = \frac{30}{60}.$$

Now we can see that $\frac{29}{60} < \frac{30}{60}$ since we are comparing 29 pieces to 30 pieces where these pieces all have the same size. So we find

$$\frac{29}{60} < \frac{1}{2}.$$

Next, to compare $\frac{1}{2}$ to $\frac{45}{88}$ we can write $\frac{1}{2}$ with a denominator of 88, multiplying numerator and denominator by 44 this time:

$$\frac{1}{2} = \frac{44 \times 1}{44 \times 2} = \frac{44}{88}.$$

We know that $\frac{44}{88} < \frac{45}{88}$ because 44 pieces is less than 45 pieces and the pieces all have the same size. So we see that

$$\frac{1}{2} < \frac{45}{88}.$$

Combining the reasoning of the two paragraphs above gives

$$\frac{29}{60} < \frac{1}{2} < \frac{45}{88}$$

and so $\frac{45}{88}$ is greater than $\frac{29}{60}$.

c. The reasoning here will be like that of parts (a) and (b) if we can identify the benchmark fraction to compare with $\frac{8}{25}$ and $\frac{19}{45}$. One possible choice for a benchmark comparison is the fraction $\frac{2}{5}$, convenient because one of our fractions has 25 as a denominator. Since $25 = 5 \times 5$, we can convert the fraction $\frac{2}{5}$ to twenty-fifths:

$$\frac{2}{5} = \frac{5 \times 2}{5 \times 5} = \frac{10}{25}.$$

Now $\frac{8}{25} < \frac{10}{25}$ because 8 is less than 10 and both fractions have a denominator of 25. So we have found that

$$\frac{8}{25} < \frac{2}{5}.$$

Since we used $\frac{2}{5}$ for comparison with $\frac{8}{25}$, we should also use $\frac{2}{5}$ for comparison with $\frac{19}{45}$. Since $45 = 9 \times 5$, we can convert the fraction $\frac{2}{5}$ to forty-fifths:

$$\frac{2}{5} = \frac{9 \times 2}{9 \times 5} = \frac{18}{45}.$$

Now $\frac{18}{45} < \frac{19}{45}$ because 18 is less than 19 and both fractions have a denominator of 45. So we have found that

$$\frac{2}{5} < \frac{19}{45}.$$

Combining the previous work, we see that

$$\frac{8}{25} < \frac{2}{5} < \frac{19}{45}.$$

Since $8 \times 3 = 24$, we have

$$\frac{1}{3} = \frac{8 \times 1}{8 \times 3} = \frac{8}{24}.$$

This is close to $\frac{8}{25}$ and this was what motivated the choice of $\frac{1}{3}$ (we will see below that $\frac{19}{45}$ is also close to $\frac{1}{3}$, making $\frac{1}{3}$ an appropriate fraction for comparison). To see which is larger, $\frac{1}{3}$ or $\frac{8}{25}$, note that $\frac{1}{25} < \frac{1}{24}$ because if a whole is broken into 24 equal sized pieces these pieces will be larger than if the same whole is broken into 25 equal sized pieces. So we can conclude that $\frac{8}{25} < \frac{8}{24}$ giving

$$\frac{8}{25} < \frac{1}{3}.$$

Since we used $\frac{1}{3}$ for comparison with $\frac{8}{25}$ we should also use $\frac{1}{3}$ for comparison with $\frac{19}{45}$. Since $45 = 15 \times 3$, we can convert the fraction $\frac{1}{3}$ to forty-fifths:

$$\frac{1}{3} = \frac{15 \times 1}{15 \times 3} = \frac{15}{45}.$$

Now $\frac{15}{45} < \frac{19}{45}$ because 15 is less than 19 and both fractions have a denominator of 45. So we have found that

$$\frac{1}{3} < \frac{19}{45}.$$

Combining the work of the previous two paragraphs we see that

$$\frac{8}{25} < \frac{1}{3} < \frac{19}{45}.$$

The key to using this method for comparing fractions is identifying a benchmark fraction for comparison. This requires either a good number sense or a lot of experience.

4.NF Using Benchmarks to Compare Fractions
Typeset April 7, 2015 at 18:34:20. Licensed by Illustrative Mathematics under a
Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.