Illustrative Mathematics

8-NS Calculating and Rounding Numbers

Alignments to Content Standards: 8.NS.A

Task

Lucia's calculator gives a value of

$$3.14159265$$

for π.

a. Is the equation

$$\pi = 3.14159265$$

valid? Explain.

b. When Lucia computes π^2 on her calculator, using the π and square buttons, it shows

$$9.86960440.$$

On the other hand, when she calculates $(3.14159265)^2$ on her calculator it shows

$$9.86960438.$$

Explain why the calculator shows different answers for what appears to be the same quantity.

IM Commentary

Calculators are tremendously useful tools but sometimes they can mislead. Whenever dealing with an irrational number, such as π, the calculator makes an appropriate approximation. For practical purposes this is sufficient. It is important, however, when performing arithmetic operations, to be careful about the effects of working with rounded numbers. The approximation 3.14159265 for π is a little too small, by a little more than three billionths. On the other hand, when asked to find π^2 the calculator has a built in algorithm which finds the correct answer to the nearest billionth. Calculators provide many opportunities, when appropriately studied, for understanding the mathematics behind its calculations.

This task is intended for instructional (rather than assessment) purposes, providing an opportunity to discuss technology as it relates to irrational numbers and calculations in general. The task gives a concrete example where rounding and then multiplying does not yield the same answer as multiplying and then rounding. The teacher may wish to remind students of the mathematical symbols usually used where Lucia used "=." The symbols $\sim$ and $\approx$ are both used to mean "is approximately equal to" so Lucia could write $\pi \approx 3.14159265$ and this would be correct.

There are many variants on this theme if the teacher wishes to work with an expression other than π. For example, a calculator will find, to eight decimal places, that

$$\sqrt{2} \approx 1.41421356.$$

If the student copies this number down and then enters it on the calculator and squares it, the calculator will show 1.99999999 rather than 2. This also gives some insight into how a calculator works because if the expression on the screen, after calculating $\sqrt{2}$, is squared directly, the calculator will show 2: in other words, the calculator has stored in its memory, in this case, more information than is shown on the screen!

The teacher may also wish to discuss with students the fact that this issue comes up with fractions as well. If the students compute $\frac{1}{17}$ on a calculator they will find 0.05882353 (or perhaps a few extra digits) but this can not be correct because the decimal expansion of $\frac{1}{17}$ is not terminating.

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Solution

a. The equation

$$\pi = 3.14159265$$

is not valid. The number π is irrational while 3.14159265 is a rational number. As an irrational number, the decimal expansion of π never repeats and 3.14159265 is the best approximation with 8 decimal places. To thirty places, the best approximation of π is

$$3.141592653589793238462643383279.$$

For numbers like π, e, and $\sqrt{2}$ which have explicit symbols to represent them, it is best to use the symbol rather than converting to a rounded off decimal. In other words, Lucia can just leave the π without trying to convert to a decimal. In computational contexts where what is sought is not an exact expression but an approximate answer, then it would be appropriate and even necessary to write

$$\pi \approx 3.14159265$$

using the approximately equal to sign $\approx$ instead of the equals sign $=$.

b. When asked to perform the calculation of $(3.14159265)^2$ the calculator will find this quantity and round to the nearest hundred millionth since the display only allows to show the answer to the nearest hundred millionth. On the other hand, when asked to find π^2, rather than taking an approximate value for π and then squaring, the calculator has a built in algorithm which can find the quantity π^2 to the desired level of precision, in this case to the nearest hundred millionth.

The fact that the calculator reads different numbers for π^2 and for $(3.14159265)^2$ indirectly answers (a): since these quantities are not equal, it cannot be that $\pi = 3.14159265$.